



On similarity and inclusion measures between type-2 fuzzy sets with an application to clustering

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ABSTRACT

In this paper we define similarity and inclusion measures between type-2 fuzzy sets. We then discuss their properties and also consider the relationships between them. Several examples are used to present the calculation of these similarity and inclusion measures between type-2 fuzzy sets. We finally combine the proposed similarity measures with Yang and Shih's [M.S. Yang, H.M. Shih, Cluster analysis based on fuzzy relations, *Fuzzy Sets and Systems* 120 (2001) 197–212] algorithm as a clustering method for type-2 fuzzy data. These clustering results are compared with Hung and Yang's [W.L. Hung, M.S. Yang, Similarity measures between type-2 fuzzy sets, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 12 (2004) 827–841] results. According to different α -level, these clustering results consist of a better hierarchical tree.

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1. Introduction

The concept of type-2 fuzzy sets was first proposed by Zadeh [1] as an extension of a type-1 fuzzy set, i.e. an ordinary fuzzy set (Zadeh [2]). More studies on type-2 fuzzy sets were then sequentially explored by Mizumoto and Tanaka [3,4], Nieminen [5] and Yager [6]. However, type-2 fuzzy sets had been ignored for a while because they were relatively hard to understand and clarify compared with fuzzy sets. Recently, they have attracted more and more attention from researchers and been analyzed and discussed in advance. Type-2 fuzzy sets have been widely applied to areas such as decision theory [6], signal processing [7], speech recognition [8], transport scheduling [9], pattern recognition [10], correlation coefficient [11], forecasting of time series [12], fuzzy equation systems [13], and so forth.

Type-2 fuzzy sets can improve certain kinds of inference better than do fuzzy sets with increasing imprecision, uncertainty and fuzziness in information. A type-2 fuzzy set is an extension to a fuzzy set in which its membership function falls into a fuzzy set in the interval $[0, 1]$. More algebraic operations on type-2 fuzzy sets had been conducted by Dubois and Prade [14,15], Mizumoto and Tanaka [3,4], Karnik and Mendel [16], and Tahayori et al. [17]. Furthermore, Mendel and John [18] proposed a new representation theorem that can use and explain the union, intersection and complement operations for type-2 fuzzy sets with enhanced easiness without the necessity of using the complicated extension principle.

As an important tool for determining the similarity between two objects, Zadeh [19] initiated fuzzy similarity measure, and later on, various similarity measures for fuzzy sets have been sequentially proposed. Pappis and Karacapilidis [20] proposed three similarity measures based on union and intersection operations, the maximum difference, and the difference and sum of membership grades. Liu [21] provided the axiom definition and properties of similarity measures between fuzzy sets. Turksen and Zhong [22] applied similarity measures between fuzzy sets for an approximate analogical reasoning.

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Buckley and Hayashi [23] used a similarity measure between fuzzy sets to determine whether a rule should be made for rule matching in fuzzy control and neural networks. In a multimedial database query, Candan et al. [24] applied similarity measures to develop query processing with different fuzzy semantics.

When Zadeh [2] introduced fuzzy sets, he also defined the inclusion for fuzzy sets. Afterwards, the inclusion measure for fuzzy sets as to define the degree to which a fuzzy set is included in another fuzzy set had been studied in the literature. Sinha and Dogherty [25] analyzed inclusion measures for general fuzzy sets based on particular axiom definitions. Chatzis and Pitas [26] proposed a new fuzzy inclusion indicator for morphological operations. Kehagins and Konstantinidou [27] introduced L-fuzzy valued inclusion measures and then explored the relationships between inclusion measures and fuzzy distance among general fuzzy sets. Cornelis et al. [28] revisited the Sinha and Dogherty approach by exposing it in a clearer way. Zeng and Li [29] investigated the relations among inclusion measures, similarity measures, and the fuzziness of fuzzy sets.

Little effort has been made as to the similarity and inclusion measures for type-2 fuzzy sets. Hence, new similarity and inclusion measures between type-2 fuzzy sets are proposed in this paper. For practical reasons, we would explain similarity measures between Gaussian type-2 fuzzy sets by examples. We then combine the proposed similarity measures with Yang and Shih's [30] algorithm for clustering type-2 fuzzy data. The clustering results are logically expressed in a hierarchical tree structure. The remainder of this paper is organized as follows. In Section 2, definitions and properties concerned with the similarity and inclusion measures between fuzzy sets will be first reviewed. A brief review for type-2 fuzzy sets is then given. In Section 3, new definitions and relevant properties with respect to inclusion and similarity measures between type-2 fuzzy sets will be proposed and discussed. Some examples and comparisons will be presented in Section 4. The combined clustering method with Yang and Shih's [30] algorithm as a hierarchical clustering for type-2 fuzzy data is also considered. Finally, conclusions will be stated in Section 5.

2. Preliminaries

In this section, we first discuss and review similarity and inclusion measures between fuzzy sets. We then give some definitions and notations for convenience of explaining general concepts concerned with type-2 fuzzy sets.

2.1. Similarity and inclusion measures between fuzzy sets

Zadeh [2] initiated fuzzy sets which describe everything as a matter of degree and can be used to capture the uncertainty in imprecision and vagueness in a mathematical way. On the other hand, a similarity between fuzzy sets is an important way to measure the degree of similarity between two fuzzy concepts. Zwick et al. [31] reviewed and compared 19 similarity measures between fuzzy sets based on both geometric and set-theoretic ways. Pappis and Karacapilidis [20] introduced three similarity measures between fuzzy sets. After that, some researchers (see [22,32,33]) gave more similarity measures of fuzzy sets.

Throughout this paper, the following notations are used. X is the universe of discourse; $F_1(X)$ is the class of all fuzzy sets of X ; $\mu_A : X \rightarrow [0, 1]$ is the membership function of A in $F_1(X)$; Consider two fuzzy sets A and B in $F_1(X)$, we call $S(A, B)$ the similarity measure between A and B , if the mapping $S : F_1(X) \times F_1(X) \rightarrow [0, 1]$ satisfies the following axioms (see Liu [21]):

- (S1) $S(A, B) = S(B, A), \forall A, B \in F_1(X)$;
- (S2) $S(D, D^c) = 0, \forall D \in \mathcal{P}(X)$ (the power set of X);
- (S3) $S(E, E) = \max_{A, B \in F_1(X)} S(A, B), \forall E \in F_1(X)$;
- (S4) $\forall A, B, C \in F_1(X)$, if $A \subset B \subset C$, then $S(A, B) \geq S(A, C)$ and $S(B, C) \geq S(A, C)$.

Pappis and Karacapilidis [20] proposed the following three similarity measures for the finite set $X = \{x_1, x_2, x_3, \dots, x_n\}$:

$$S_1(A, B) = \frac{\sum_{x \in X} \min\{\mu_A(x), \mu_B(x)\}}{\sum_{x \in X} \max\{\mu_A(x), \mu_B(x)\}} \quad (1)$$

$$S_2(A, B) = 1 - \max_{x \in X} |\mu_A(x) - \mu_B(x)| \quad (2)$$

$$S_3(A, B) = 1 - \frac{\sum_{x \in X} |\mu_A(x) - \mu_B(x)|}{\sum_{x \in X} |\mu_A(x) + \mu_B(x)|}. \quad (3)$$

Based on the three similarity measures of (1)–(3), the authors [21,22,32], had given more similarities between fuzzy sets. In the next section, we will propose a new similarity measure between type-2 fuzzy sets and then apply it for clustering type-2 fuzzy data.

Consider two fuzzy sets A and B in $F_1(X)$. Zadeh [2] gave a definition of fuzzy set inclusion with: $A \subseteq B \iff \mu_A(x) \leq \mu_B(x), \forall x \in X$. Sinha and Dogherty [25] considered an indicator $I(A, B)$ for an inclusion measure between two fuzzy sets A and B and then gave several axioms that $I(A, B)$ needs to satisfy (also see [27–29,34]). Here, we adopt a simple definition of

the inclusion measure from Zeng and Li [29] as follows. $I(A, B)$ is called an inclusion measure of fuzzy sets A and B in $F_1(X)$, if the mapping $I : F_1(X) \times F_1(X) \rightarrow [0, 1]$ satisfies the following axioms:

$$(I1) \ I(A, \phi) = 0;$$

$$(I2) \ I(A, B) = 1 \iff A \subseteq B;$$

$$(I3) \ \text{For any } A, B, C \in F_1(X), \text{ if } A \subseteq B \subseteq C, \text{ then } I(C, A) \leq I(B, A), I(C, A) \leq I(C, B).$$

The following inclusion measures for the finite set $X = \{x_1, x_2, x_3, \dots, x_n\}$ were proposed by Shinha and Dougherty [25], Bloch and Maitre [35] and Werman and Peleg [36].

$$I_1(A, B) = \frac{\sum_{x \in X} \min\{\mu_A(x), \mu_B(x)\}}{\sum_{x \in X} \mu_A(x)} \quad (4)$$

$$I_2(A, B) = \inf_x \max\{\mu_B(x), 1 - \mu_A(x)\} \quad (5)$$

$$I_3(A, B) = \inf_x \min\{1, \lambda(\mu_A(x)) + \lambda(1 - \mu_B(x))\} \quad (6)$$

$$I_4(A, B) = \int_0^1 \inf_{x \in A^\alpha} \mu_B(x) d\alpha. \quad (7)$$

After that, more researches about inclusion measures of fuzzy sets had been investigated (see [29,34]).

2.2. Type-2 fuzzy sets

In this part, we give the relative definitions and notations of type-2 fuzzy sets with its properties based on Mendel and John [18] where they presented a new representation of type-2 fuzzy sets to help us specify the type-2 fuzzy sets into a simple way.

Definition 1 (Mendel and John [18]). A type-2 fuzzy set, denoted \tilde{A} , is characterized by a type-2 membership function $\mu_{\tilde{A}}(x, u)$, where $x \in X$ and $u \in J_x \subseteq [0, 1]$, i.e.,

$$\tilde{A} = \{(x, u), \mu_{\tilde{A}}(x, u) | \forall x \in X, \forall u \in J_x \subseteq [0, 1]\} \quad (8)$$

in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$. \tilde{A} can be also expressed as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u), \quad J_x \subseteq [0, 1] \quad (9)$$

where \int denotes union over all admissible x and u . For discrete universe of discourse \int is replaced by Σ .

Definition 2 (Mendel and John [18]). At each value of x , say $x = x'$, the 2-D plane whose axes are u and $\mu_{\tilde{A}}(x', u)$ is called a vertical slice of $\mu_{\tilde{A}}(x, u)$. A secondary membership function is a vertical slice of $\mu_{\tilde{A}}(x, u)$. It is $\mu_{\tilde{A}}(x = x', u)$ for $x \in X$ and $\forall u \in J_{x'} \subseteq [0, 1]$, i.e.,

$$\mu_{\tilde{A}}(x = x', u) \equiv \mu_{\tilde{A}}(x') = \int_{u \in J_{x'}} f_{x'}(u) / u, \quad J_{x'} \subseteq [0, 1] \quad (10)$$

in which $0 \leq f_{x'}(u) \leq 1$.

\tilde{A} can be also re-expressed as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | \forall x \in X\} \quad (11)$$

or, as

$$\tilde{A} = \int_{x \in X} \mu_{\tilde{A}}(x) / x = \int_{x \in X} \left[\int_{u \in J_x} f_x(u) / u \right] / x, \quad J_x \subseteq [0, 1]. \quad (12)$$

Definition 3 (Mendel and John [18]). The domain of a secondary membership function is called the primary membership of x . In (12), J_x is the primary membership of x , where $J_x \subseteq [0, 1] \forall x \in X$.

Definition 4 (Mendel and John [18]). Uncertainty in the primary memberships of a type-2 fuzzy set, \tilde{A} , consists of a bounded region that we call the footprint of uncertainty (FOU). It is the union of all primary memberships, i.e.,

$$FOU(\tilde{A}) = \bigcup_{x \in X} J_x.$$

3. Proposed similarity and inclusion measures between type-2 fuzzy sets

In this section, we will propose new similarity and inclusion measures between type-2 fuzzy sets, and then discuss the relevant properties between them. The following notations are used; X is the universe of discourse; $\mathcal{F}_2(X)$ is the class of all type-2 fuzzy sets in X ; $\mu_{\tilde{A}}(x, u) : (x, u) \rightarrow [0, 1]$ is the membership function of \tilde{A} in $\mathcal{F}_2(X)$, $\forall x \in X, u \in J_x \subseteq [0, 1]$; For any $x \in X$, $f_x(u)$ is defined as $\mu_{\tilde{A}}(x, u)$, $\forall u \in J_x \subseteq [0, 1]$.

For type-2 fuzzy sets \tilde{A} and \tilde{B} , the secondary membership functions $f_x(u) = \mu_{\tilde{A}}(x, u)$ and $g_x(u) = \mu_{\tilde{B}}(x, u)$ and the footprints of uncertainty $FOU(\tilde{A})$ and $FOU(\tilde{B})$ are two key representations. We know that if $\tilde{A} \subset \tilde{B}$, then $FOU(\tilde{A}) \subset FOU(\tilde{B})$ should be true, but its inverse is not always true. Moreover, if $0 \leq f_x(u) \leq g_x(u) \leq 1, \forall x \in X, u \in J_x \subseteq [0, 1]$, then $FOU(\tilde{A}) \subset FOU(\tilde{B})$. We mention that for any $\tilde{A}, \tilde{B} \in \mathcal{F}_2(X)$, Mizumoto and Tanaka [3] defined $\tilde{A} \subset \tilde{B}$ to be $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \forall x \in X$ along with the definition of Zadeh [2]. In this sense, we can give an inclusion definition for type-2 fuzzy sets as follows.

Definition 5. For any $\tilde{A}, \tilde{B} \in \mathcal{F}_2(X)$, $\tilde{A} \subseteq \tilde{B}$ is defined as $0 \leq f_x(u) \leq g_x(u) \leq 1, \forall x \in X, u \in J_x \subseteq [0, 1]$.

Thus, by similarly following the definition of an inclusion measure I for fuzzy sets described in Section 2.1, we can give the definition of an inclusion measure for type-2 fuzzy sets as follows.

Definition 6. A real function $M : \mathcal{F}_2(X) \times \mathcal{F}_2(X) \rightarrow [0, 1]$ is called an inclusion measure, if M satisfies the following axioms:

- (I1') $M(\tilde{A}, \tilde{A}) = 1$.
- (I2') $\tilde{A} \subseteq \tilde{B} \iff M(\tilde{A}, \tilde{B}) = 1$.
- (I3') For any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}_2(X)$, if $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $M(\tilde{C}, \tilde{A}) \leq M(\tilde{B}, \tilde{A}), M(\tilde{C}, \tilde{A}) \leq M(\tilde{C}, \tilde{B})$.

Similarly, we give a similarity definition for type-2 fuzzy sets as follows.

Definition 7. A real function $N : \mathcal{F}_2(X) \times \mathcal{F}_2(X) \rightarrow [0, 1]$ is called a similarity measure, if N satisfies the following axioms:

- (S1') $N(\tilde{A}, \tilde{B}) = N(\tilde{B}, \tilde{A}), \forall \tilde{A}, \tilde{B} \in \mathcal{F}_2(X)$.
- (S2') $N(\tilde{D}, \tilde{D}^c) = 0, \forall \tilde{D} \in \mathcal{P}(X)$ (the power set of X);
- (S3') $N(\tilde{E}, \tilde{E}) = \max_{\tilde{A}, \tilde{B} \in \mathcal{F}_2(X)} N(\tilde{A}, \tilde{B}), \forall \tilde{E} \in \mathcal{F}_2(X)$.
- (S4') For any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}_2(X)$, if $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $N(\tilde{A}, \tilde{B}) \geq N(\tilde{A}, \tilde{C})$ and $N(\tilde{B}, \tilde{C}) \geq N(\tilde{A}, \tilde{C})$.

To compute the inclusion degree between type-2 fuzzy sets, we may follow a similar formula as the inclusion measure $I_1(A, B)$ of Eq. (4) defined in Section 2.1. For type-2 fuzzy sets, we need to consider the FOU of the primary membership function and also the secondary membership function. Based on this consideration, we propose a new inclusion measure between type-2 fuzzy sets \tilde{A} and \tilde{B} as follows:

$$I(\tilde{A}, \tilde{B}) = \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \min\{u \cdot f_x(u), u \cdot g_x(u)\} du}{\int_{u \in J_x} \{u \cdot f_x(u)\} du} dx \quad (13)$$

where the notation \int in Eq. (13) is an integral. For discrete universes of discourse, \int is replaced by the summation Σ .

We next propose a new similarity measure between type-2 fuzzy sets, and discuss the relevant properties between them. By considering both of the FOU of the primary membership function and the secondary membership function into the existing similarity $S_1(A, B)$ of Eq. (1) described in Section 2.1, we propose a similarity measure between type-2 fuzzy sets \tilde{A} and \tilde{B} as:

$$S(\tilde{A}, \tilde{B}) = \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \min\{u \cdot f_x(u), u \cdot g_x(u)\} du}{\int_{u \in J_x} \max\{u \cdot f_x(u), u \cdot g_x(u)\} du} dx \quad (14)$$

where the notation \int in Eq. (14) is an integral. For discrete universes of discourse, \int is replaced by the summation Σ .

Property 1. $I(\cdot, \cdot)$ is an inclusion measure on $\mathcal{F}_2(X)$.

Proof. (I1') $I(\tilde{A}, \tilde{A}) = \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \min\{u \cdot f_x(u), u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot f_x(u)\} du} dx = \frac{1}{\int_{x \in X} dx} \int_{x \in X} dx = 1$.

(I2')

$$\begin{aligned} \tilde{A} \subseteq \tilde{B} &\Rightarrow 0 \leq f_x(u) \leq g_x(u) \leq 1, \quad \forall x \in X, u \in J_x \subseteq [0, 1] \\ &\Rightarrow I(\tilde{A}, \tilde{B}) = \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \min\{u \cdot f_x(u), u \cdot g_x(u)\} du}{\int_{u \in J_x} \{u \cdot f_x(u)\} du} dx \\ &= \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \{u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot f_x(u)\} du} dx = 1. \end{aligned}$$

On the other hand, if $\tilde{A} \not\subseteq \tilde{B}$, but $\tilde{A} \cap \tilde{B} \neq \Phi$ then for any $x \in X$, there exist U_1 and U_2 with $U_1 \neq \Phi$ and $U_1 \cup U_2 = J_x$ such that $0 \leq g_x(u_1) < f_x(u_1) \leq 1, \forall u_1 \in U_1$ and $0 \leq f_x(u_2) < g_x(u_2) \leq 1, \forall u_2 \in U_2$. Then

$$I(\tilde{A}, \tilde{B}) = \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u_1 \in U_1} \min\{u_1 \cdot f_x(u_1), u_1 \cdot g_x(u_1)\} du_1 + \int_{u_2 \in U_2} \min\{u_2 \cdot f_x(u_2), u_2 \cdot g_x(u_2)\} du_2}{\int_{u \in J_x} \{u \cdot f_x(u)\} du} dx < 1.$$

However, $0 < I(\tilde{A}, \tilde{B}) \leq 1$ for $\tilde{A} \cap \tilde{B} \neq \Phi$. Thus, $I(\tilde{A}, \tilde{B}) = 1 \Rightarrow \tilde{A} \subseteq \tilde{B}$.

(13') $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \Rightarrow 0 \leq f_x(u) \leq g_x(u) \leq h_x(u) \leq 1$.

We have that

$$\begin{aligned} I(\tilde{C}, \tilde{A}) &= \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \min\{u \cdot h_x(u), u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot h_x(u)\} du} dx \\ &= \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \{u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot h_x(u)\} du} dx \\ &\leq \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \{u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot g_x(u)\} du} dx \\ &= \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \min\{u \cdot g_x(u), u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot g_x(u)\} du} dx \\ &= I(\tilde{B}, \tilde{A}). \end{aligned}$$

Similarly, we have that $I(\tilde{C}, \tilde{A}) \leq I(\tilde{C}, \tilde{B})$. \square

Property 2. (1) $\tilde{B} \subseteq \tilde{C} \Rightarrow I(\tilde{A}, \tilde{B}) \leq I(\tilde{A}, \tilde{C}), \forall \tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}_2(X)$.

(2) $\tilde{B} \subseteq \tilde{C} \Rightarrow I(\tilde{C}, \tilde{A}) \leq I(\tilde{B}, \tilde{A}), \forall \tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}_2(X)$.

Proof. (1) (i) $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \Rightarrow I(\tilde{A}, \tilde{B}) = 1$ and $I(\tilde{A}, \tilde{C}) = 1 \Rightarrow I(\tilde{A}, \tilde{B}) \leq I(\tilde{A}, \tilde{C})$.

(ii) $\tilde{B} \subseteq \tilde{A} \subseteq \tilde{C} \Rightarrow 0 \leq I(\tilde{A}, \tilde{B}) \leq 1$ and $I(\tilde{A}, \tilde{C}) = 1 \Rightarrow I(\tilde{A}, \tilde{B}) \leq I(\tilde{A}, \tilde{C})$.

(iii)

$$\begin{aligned} \tilde{B} \subseteq \tilde{C} \subseteq \tilde{A} &\Rightarrow 0 \leq g_x(u) \leq h_x(u) \leq f_x(u) \leq 1, \quad \forall x \in X, u \in J_x \subseteq [0, 1] \\ &\Rightarrow \min\{u \cdot f_x(u), u \cdot g_x(u)\} \leq \min\{u \cdot f_x(u), u \cdot h_x(u)\}. \end{aligned}$$

$$\begin{aligned} I(\tilde{A}, \tilde{B}) &= \frac{1}{\int_{x \in X} dx} \int_x \frac{\int_{u \in J_x} \min\{u \cdot f_x(u), u \cdot g_x(u)\} du}{\int_{u \in J_x} \{u \cdot f_x(u)\} du} dx \\ &\leq \frac{1}{\int_{x \in X} dx} \int_x \frac{\int_{u \in J_x} \min\{u \cdot f_x(u), u \cdot h_x(u)\} du}{\int_{u \in J_x} \{u \cdot f_x(u)\} du} dx \\ &= I(\tilde{A}, \tilde{C}). \end{aligned}$$

(2) (i)

$$\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \Rightarrow \frac{u \cdot f_x(u)}{u \cdot h_x(u)} \leq \frac{u \cdot f_x(u)}{u \cdot g_x(u)}.$$

$$\begin{aligned} I(\tilde{C}, \tilde{A}) &= \frac{1}{\int_{x \in X} dx} \int_x \frac{\int_{u \in J_x} \min\{u \cdot h_x(u), u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot h_x(u)\} du} dx \\ &\leq \frac{1}{\int_{x \in X} dx} \int_x \frac{\int_{u \in J_x} \min\{u \cdot g_x(u), u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot g_x(u)\} du} dx \\ &= I(\tilde{B}, \tilde{A}). \end{aligned}$$

(ii) $\tilde{B} \subseteq \tilde{A} \subseteq \tilde{C} \Rightarrow 0 \leq I(\tilde{C}, \tilde{A}) \leq 1$ and $I(\tilde{B}, \tilde{A}) = 1 \Rightarrow I(\tilde{C}, \tilde{A}) \leq I(\tilde{B}, \tilde{A})$.

(iii) $\tilde{B} \subseteq \tilde{C} \subseteq \tilde{A} \Rightarrow 0 \leq I(\tilde{C}, \tilde{A}) = 1$ and $I(\tilde{B}, \tilde{A}) = 1 \Rightarrow I(\tilde{C}, \tilde{A}) \leq I(\tilde{B}, \tilde{A})$. \square

If we define the union (\cup) for any type-2 fuzzy sets \tilde{A} and \tilde{B} on $\mathcal{F}_2(X)$ with the maximum operation, and define the intersection (\cap) with the minimum operation, we will have the following results.

Property 3. For any type-2 fuzzy sets $\tilde{A}, \tilde{B}, \tilde{C}$ on $\mathcal{F}_2(X)$, we have

- (1) $I(\tilde{A} \cup \tilde{B}, \tilde{C}) = \min\{I(\tilde{A}, \tilde{C}), I(\tilde{B}, \tilde{C})\};$
- (2) $I(\tilde{A} \cap \tilde{B}, \tilde{C}) = \max\{I(\tilde{A}, \tilde{C}), I(\tilde{B}, \tilde{C})\};$
- (3) $I(\tilde{A}, \tilde{B} \cup \tilde{C}) = \max\{I(\tilde{A}, \tilde{B}), I(\tilde{A}, \tilde{C})\};$
- (4) $I(\tilde{A}, \tilde{B} \cap \tilde{C}) = \min\{I(\tilde{A}, \tilde{B}), I(\tilde{A}, \tilde{C})\}.$

Proof. (1) We consider the following six cases:

- (i) $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \Rightarrow I(\tilde{A}, \tilde{C}) = 1, I(\tilde{B}, \tilde{C}) = 1, \tilde{A} \cup \tilde{B} = \tilde{B}$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = I(\tilde{B}, \tilde{C}) = 1$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = \min\{I(\tilde{A}, \tilde{C}), I(\tilde{B}, \tilde{C})\}.$
- (ii) $\tilde{B} \subseteq \tilde{A} \subseteq \tilde{C} \Rightarrow I(\tilde{A}, \tilde{C}) = 1, I(\tilde{B}, \tilde{C}) = 1, \tilde{A} \cup \tilde{B} = \tilde{A}$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = I(\tilde{A}, \tilde{C}) = 1$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = \min\{I(\tilde{A}, \tilde{C}), I(\tilde{B}, \tilde{C})\}.$
- (iii) $\tilde{A} \subseteq \tilde{C} \subseteq \tilde{B} \Rightarrow I(\tilde{A}, \tilde{C}) = 1, 0 \leq I(\tilde{B}, \tilde{C}) < 1, \tilde{A} \cup \tilde{B} = \tilde{B}$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = I(\tilde{B}, \tilde{C})$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = \min\{I(\tilde{A}, \tilde{C}), I(\tilde{B}, \tilde{C})\}.$
- (iv) $\tilde{C} \subseteq \tilde{A} \subseteq \tilde{B} \Rightarrow 0 \leq I(\tilde{A}, \tilde{C}) < 1, 0 \leq I(\tilde{B}, \tilde{C}) < 1, \tilde{A} \cup \tilde{B} = \tilde{B}$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = I(\tilde{B}, \tilde{C}), I(\tilde{B}, \tilde{C}) \leq I(\tilde{A}, \tilde{C})$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = \min\{I(\tilde{A}, \tilde{C}), I(\tilde{B}, \tilde{C})\}.$
- (v) $\tilde{B} \subseteq \tilde{C} \subseteq \tilde{A} \Rightarrow I(\tilde{B}, \tilde{C}) = 1, 0 \leq I(\tilde{A}, \tilde{C}) < 1, \tilde{A} \cup \tilde{B} = \tilde{A}$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = I(\tilde{A}, \tilde{C}), I(\tilde{A}, \tilde{C}) \leq I(\tilde{B}, \tilde{C})$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = \min\{I(\tilde{A}, \tilde{C}), I(\tilde{B}, \tilde{C})\}.$
- (vi) $\tilde{C} \subseteq \tilde{B} \subseteq \tilde{A} \Rightarrow 0 \leq I(\tilde{A}, \tilde{C}) < 1, 0 \leq I(\tilde{B}, \tilde{C}) < 1, \tilde{A} \cup \tilde{B} = \tilde{A}$
 $\Rightarrow I(\tilde{A} \cup \tilde{B}, \tilde{C}) = \min\{I(\tilde{A}, \tilde{C}), I(\tilde{B}, \tilde{C})\}.$

Similarly, we obtain (2), (3) and (4). \square

Property 4. $S(\cdot, \cdot)$ is a similarity measure on $\mathcal{F}_2(X)$.

Proof. (S1'), (S2'), (S3') are trivial.

(S4'): If $\tilde{A} \subseteq \tilde{B}$, then

$$\begin{aligned} S(\tilde{A}, \tilde{B}) &= \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \min\{u \cdot f_x(u), u \cdot g_x(u)\} du}{\int_{u \in J_x} \max\{u \cdot f_x(u), u \cdot g_x(u)\} du} dx \\ &= \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \{u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot g_x(u)\} du} dx \\ &\geq \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \{u \cdot f_x(u)\} du}{\int_{u \in J_x} \{u \cdot h_x(u)\} du} dx \\ &= \frac{1}{\int_{x \in X} dx} \int_{x \in X} \frac{\int_{u \in J_x} \min\{u \cdot f_x(u), u \cdot h_x(u)\} du}{\int_{u \in J_x} \max\{u \cdot f_x(u), u \cdot h_x(u)\} du} dx \\ &= S(\tilde{A}, \tilde{C}). \end{aligned}$$

Similarly, $S(\tilde{B}, \tilde{C}) \geq S(\tilde{A}, \tilde{C})$. \square

In the following, we present the relationship between the similarity and inclusion measures of type-2 fuzzy sets based on the definitions of Eqs. (13) and (14).

Property 5. For any two type-2 fuzzy sets \tilde{A} and \tilde{B} , let $S(\tilde{A}, \tilde{B}) = \min\{I(\tilde{A}, \tilde{B}), I(\tilde{B}, \tilde{A})\}$. If I is an inclusion measure of type-2 fuzzy sets \tilde{A} and \tilde{B} , then S is the similarity measure of type-2 fuzzy sets \tilde{A} and \tilde{B} .

Proof. (S1') $S(\tilde{A}, \tilde{B}) = \min\{I(\tilde{A}, \tilde{B}), I(\tilde{B}, \tilde{A})\} = \min\{I(\tilde{B}, \tilde{A}), I(\tilde{A}, \tilde{B})\} = S(\tilde{B}, \tilde{A})$.

(S2') $S(D, D^c) = \min\{I(D, D^c), I(D^c, D)\} = 0$ if D is a crisp set.

(S3') Since $S(\tilde{E}, \tilde{E}) = 1$ and $0 \leq I(\tilde{A}, \tilde{B}) \leq 1, 0 \leq I(\tilde{B}, \tilde{A}) \leq 1$,

$$S(\tilde{E}, \tilde{E}) = \max_{\tilde{A}, \tilde{B} \in \mathcal{F}_2(X)} S(\tilde{A}, \tilde{B}), \quad \forall \tilde{E} \in \mathcal{F}_2(X).$$

(S4') For all $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}_2(X)$,

$$\begin{aligned}\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} &\Rightarrow I(\tilde{A}, \tilde{B}) = 1, I(\tilde{A}, \tilde{C}) = 1 \\ &\Rightarrow S(\tilde{A}, \tilde{C}) = \min \{I(\tilde{A}, \tilde{C}), I(\tilde{C}, \tilde{A})\} = I(\tilde{C}, \tilde{A}) \\ &\text{and } S(\tilde{A}, \tilde{B}) = \min \{I(\tilde{A}, \tilde{B}), I(\tilde{B}, \tilde{A})\} = I(\tilde{B}, \tilde{A}).\end{aligned}$$

Thus, $I(\tilde{C}, \tilde{A}) \leq I(\tilde{B}, \tilde{A}) \Rightarrow S(\tilde{A}, \tilde{C}) \leq S(\tilde{A}, \tilde{B})$.

Similarly, we have $S(\tilde{A}, \tilde{C}) \leq S(\tilde{B}, \tilde{C})$. \square

Property 6. For any two type-2 fuzzy sets \tilde{A} and \tilde{B} , let $I(\tilde{A}, \tilde{B}) = S(\tilde{A}, \tilde{A} \cap \tilde{B})$. If S is a similarity measure of type-2 fuzzy sets \tilde{A} and \tilde{B} , then I is an inclusion measure of type-2 fuzzy sets \tilde{A} and \tilde{B} .

Proof. (I1') $I(\tilde{A}, \tilde{A}) = S(\tilde{A}, \tilde{A} \cap \tilde{A}) = S(\tilde{A}, \tilde{A}) = 1$.

(I2')

$$\begin{aligned}\tilde{A} \subseteq \tilde{B} &\Rightarrow I(\tilde{A}, \tilde{B}) = S(\tilde{A}, \tilde{A} \cap \tilde{B}) = S(\tilde{A}, \tilde{A}) = 1. \\ I(\tilde{A}, \tilde{B}) = 1 &\Rightarrow 1 = I(\tilde{A}, \tilde{B}) = S(\tilde{A}, \tilde{A} \cap \tilde{B}) = S(\tilde{A}, \tilde{A}) \\ &\Rightarrow \tilde{A} = \tilde{A} \cap \tilde{B} \\ &\Rightarrow \tilde{A} \subseteq \tilde{B}.\end{aligned}$$

(I3') For $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, we have $I(\tilde{C}, \tilde{A}) = S(\tilde{C}, \tilde{C} \cap \tilde{A}) = S(\tilde{C}, \tilde{A})$ and $I(\tilde{B}, \tilde{A}) = S(\tilde{B}, \tilde{B} \cap \tilde{A}) = S(\tilde{B}, \tilde{A})$.

By the similarity definition of Eq. (14), because $S(\tilde{C}, \tilde{A}) \leq S(\tilde{B}, \tilde{A})$, we have $I(\tilde{C}, \tilde{A}) \leq I(\tilde{B}, \tilde{A})$. Similarly, $I(\tilde{C}, \tilde{B}) = S(\tilde{C}, \tilde{C} \cap \tilde{B}) = S(\tilde{C}, \tilde{B})$. Because $S(\tilde{C}, \tilde{A}) \leq S(\tilde{C}, \tilde{B})$, we have $I(\tilde{C}, \tilde{A}) \leq I(\tilde{C}, \tilde{B})$. \square

4. Examples and comparisons

In this section, we present some examples to demonstrate the similarity and inclusion measures between type-2 fuzzy sets. For any two type-2 fuzzy sets \tilde{A} and \tilde{B} , the similarity $S(\tilde{A}, \tilde{B})$ and inclusion degree $I(\tilde{A}, \tilde{B})$ are calculated using the proposed Eqs. (13) and (14). We will finally combine the proposed similarity measure with Yang and Shih's [30] algorithm as a clustering method for type-2 fuzzy data. In the final example, we use a data set to demonstrate clustering results and then compare these results with Hung and Yang's [37] method.

Example 1. Assume that there two patterns denoted with type-2 fuzzy sets in $X = \{x_1, x_2, x_3\}$. The two patterns are denoted as follows:

$$\tilde{A} = \{(x_i, \mu_{\tilde{A}}(x_i)) | x_i \in X = \{x_1, x_2, x_3\}\} \quad \text{and} \quad \tilde{B} = \{(x_i, \mu_{\tilde{B}}(x_i)) | x_i \in X = \{x_1, x_2, x_3\}\}$$

where

$$\begin{aligned}\mu_{\tilde{A}}(x_1) &= \{(0.1, 0.05), (0.3, 0.2), (0.5, 0.4), (0.9, 0.7)\}, \\ \mu_{\tilde{A}}(x_2) &= \{(0.2, 0.1), (0.4, 0.2), (0.6, 0.8), (0.8, 0.5)\}, \\ \mu_{\tilde{A}}(x_3) &= \{(0.2, 0.2), (0.3, 0.4), (0.6, 1.0), (0.8, 0.6)\}, \\ \mu_{\tilde{B}}(x_1) &= \{(0.1, 0.03), (0.3, 0.1), (0.5, 0.3), (0.9, 0.5)\}, \\ \mu_{\tilde{B}}(x_2) &= \{(0.2, 0.08), (0.4, 0.1), (0.6, 0.7), (0.8, 0.4)\}, \\ \mu_{\tilde{B}}(x_3) &= \{(0.2, 0.1), (0.3, 0.4), (0.6, 0.5), (0.8, 0.4)\}.\end{aligned}$$

Assume that a datum $\tilde{C} = \{(x_i, \mu_{\tilde{C}}(x_i)) | x_i \in X = \{x_1, x_2, x_3\}\}$ is given with:

$$\begin{aligned}\mu_{\tilde{C}}(x_1) &= \{(0.1, 0.1), (0.3, 0.1), (0.5, 0.3), (0.9, 0.6)\}, \\ \mu_{\tilde{C}}(x_2) &= \{(0.2, 0.2), (0.4, 0.2), (0.6, 0.5), (0.8, 0.3)\}, \\ \mu_{\tilde{C}}(x_3) &= \{(0.2, 0.1), (0.3, 0.3), (0.6, 0.4), (0.8, 0.5)\}.\end{aligned}$$

Therefore, according to Eq. (14), the similarity degree between type-2 fuzzy sets \tilde{A} and \tilde{B} is given by:

$$\begin{aligned}S(\tilde{A}, \tilde{B}) &= \frac{1}{3} \\ &\times \left\{ \frac{\min\{0.1 \times 0.05, 0.1 \times 0.03\} + \min\{0.3 \times 0.2, 0.3 \times 0.1\} + \min\{0.5 \times 0.4, 0.5 \times 0.3\} + \min\{0.9 \times 0.7, 0.9 \times 0.5\}}{\max\{0.1 \times 0.5, 0.1 \times 0.03\} + \max\{0.3 \times 0.2, 0.3 \times 0.1\} + \max\{0.5 \times 0.4, 0.5 \times 0.3\} + \max\{0.9 \times 0.7, 0.9 \times 0.5\}} \right. \\ &\quad \left. + \frac{\min\{0.2 \times 0.1, 0.2 \times 0.08\} + \min\{0.4 \times 0.2, 0.4 \times 0.1\} + \min\{0.6 \times 0.8, 0.6 \times 0.7\} + \min\{0.8 \times 0.5, 0.8 \times 0.4\}}{\max\{0.2 \times 0.1, 0.2 \times 0.08\} + \max\{0.4 \times 0.2, 0.4 \times 0.1\} + \max\{0.6 \times 0.8, 0.6 \times 0.7\} + \max\{0.8 \times 0.5, 0.8 \times 0.4\}} \right\}\end{aligned}$$

$$+ \frac{\min\{0.2 \times 0.2, 0.2 \times 0.1\} + \min\{0.3 \times 0.4, 0.3 \times 0.4\} + \min\{0.6 \times 1.0, 0.6 \times 0.5\} + \min\{0.8 \times 0.6, 0.8 \times 0.4\}}{\max\{0.2 \times 0.2, 0.2 \times 0.1\} + \max\{0.3 \times 0.4, 0.3 \times 0.4\} + \max\{0.6 \times 1.0, 0.6 \times 0.5\} + \max\{0.8 \times 0.6, 0.8 \times 0.4\}} \Big\} \\ = 0.7108.$$

Similarity, we have that $S(\tilde{A}, \tilde{C}) = 0.6831$ and $S(\tilde{B}, \tilde{C}) = 0.8264$. Based on the above calculation results, it is seen that the datum \tilde{C} is closer to the pattern \tilde{B} than \tilde{A} , according to the principle of the maximum degree of similarity. This result well matches the structure of the data set $\{\tilde{A}, \tilde{B}, \tilde{C}\}$.

Furthermore, the inclusion degree between type-2 fuzzy sets \tilde{A} and \tilde{B} , according to Eq. (13), is given by:

$$I(\tilde{A}, \tilde{B}) = \frac{1}{3} \\ \times \left\{ \frac{\min\{0.1 \times 0.05, 0.1 \times 0.03\} + \min\{0.3 \times 0.2, 0.3 \times 0.1\} + \min\{0.5 \times 0.4, 0.5 \times 0.3\} + \min\{0.9 \times 0.7, 0.9 \times 0.5\}}{0.1 \times 0.5 + 0.3 \times 0.2 + 0.5 \times 0.4 + 0.9 \times 0.7} \right. \\ + \frac{\min\{0.2 \times 0.1, 0.2 \times 0.08\} + \min\{0.4 \times 0.2, 0.4 \times 0.1\} + \min\{0.6 \times 0.8, 0.6 \times 0.7\} + \min\{0.8 \times 0.5, 0.8 \times 0.4\}}{0.2 \times 0.1 + 0.4 \times 0.2 + 0.6 \times 0.8 + 0.8 \times 0.5} \\ \left. + \frac{\min\{0.2 \times 0.2, 0.2 \times 0.1\} + \min\{0.3 \times 0.4, 0.3 \times 0.4\} + \min\{0.6 \times 1.0, 0.6 \times 0.5\} + \min\{0.8 \times 0.6, 0.8 \times 0.4\}}{0.2 \times 0.2 + 0.3 \times 0.4 + 0.6 \times 1.0 + 0.8 \times 0.6} \right\} \\ = 0.7108.$$

Similarly, we have that $I(\tilde{B}, \tilde{A}) = 1$, $I(\tilde{A}, \tilde{C}) = 0.6890$, $I(\tilde{C}, \tilde{A}) = 1$, $I(\tilde{B}, \tilde{C}) = 0.8711$ and $I(\tilde{C}, \tilde{B}) = 0.8969$. Based on the above calculations, we find that $S(\tilde{A}, \tilde{B}) = 0.7108 = \min\{0.7108, 1.0\} = \min\{I(\tilde{A}, \tilde{B}), I(\tilde{B}, \tilde{A})\}$. This result demonstrates Property 5.

Example 2. Assume that three type-2 fuzzy sets with the secondary membership functions as follows:

$$f_{x_i}(u) = \begin{cases} a_i, & u \in [0, 1], \text{ where } i = 1, 2, a_1 = 0.8, a_2 = 5. \\ 0, & \text{otherwise} \end{cases} \\ g_{x_i}(u) = \begin{cases} \frac{c_i}{b_i}u, & u \in [0, b_i], \text{ where } i = 1, 2, b_1 = 0.5, c_1 = 0.7, \\ \frac{c_i}{b_i - 1}u, & u \in [b_i, 1] \text{ and } b_2 = 0.4, c_2 = 0.6. \end{cases} \\ h_{x_i}(u) = \begin{cases} \frac{e_i}{d_i}u, & u \in [0, d_i], \text{ where } i = 1, 2, d_1 = 0.3, e_1 = 0.8, k_1 = 0.6, \\ h_i, & u \in [d_i, k_i], \text{ and } d_2 = 0.4, e_2 = 0.7, k_2 = 0.5. \\ \frac{e_i}{d_i - 1}, & u \in [k_i, 1]. \end{cases}$$

Therefore, according to Eq. (14), the similarity degree between type-2 fuzzy sets \tilde{A} and \tilde{B} is given by:

$$S(\tilde{A}, \tilde{B}) = \frac{1}{2} \left\{ \frac{\int_0^{0.5} u^{\frac{0.7}{0.5-1}} du + \int_{0.5}^1 u \cdot \frac{0.7}{0.5-1} (u-1) du}{\int_0^1 u \times 0.8 du} \right. \\ \left. + \frac{\int_0^{0.33} u^{\frac{0.6}{0.4-1}} du + \int_{0.33}^{0.5} u \times 0.5 du + \int_{0.5}^1 u^{\frac{0.6}{0.4-1}} (u-1) du}{\int_0^{0.33} u \cdot 5 du + \int_{0.33}^{0.4} u^{\frac{0.6}{0.4-1}} du + \int_{0.4}^{0.5} u^{\frac{0.6}{0.4-1}} (u-1) du + \int_{0.5}^1 u \times 0.5 du} \right\} \\ = 0.4137.$$

Similarly, we have that $S(\tilde{A}, \tilde{C}) = 0.5289$ and $S(\tilde{B}, \tilde{C}) = 0.9377$. Obviously, \tilde{C} is closer to \tilde{B} than \tilde{A} .

To explore the drawback of Hausdorff distance between type-2 fuzzy sets using α -cut method proposed by Hung and Yang [37], we use the following example for explanation and comparison.

Example 3. Assume that there are two type-2 fuzzy patterns in $X = \{x\}$ denoted as follows:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\} \quad \text{and} \quad \tilde{B} = \{(x, \mu_{\tilde{B}}(x))\},$$

where

$$\mu_{\tilde{A}}(x) = \{(0.2, 0.65), (0.4, 0.8), (0.5, 0.6), (0.6, 0.5), (0.8, 0.2)\}$$

$$\mu_{\tilde{B}}(x) = \{(0.2, 0.65), (0.4, 0.8), (0.5, 0.9), (0.6, 0.7), (0.8, 0.4)\}.$$

Assumed that a sample pattern $\tilde{C} = \{(x, \mu_{\tilde{C}}(x))\}$ is given, where:

$$\mu_{\tilde{C}}(x) = \{(0.2, 0.6), (0.4, 0.7), (0.5, 0.75), (0.6, 0.9), (0.8, 0.4)\}.$$

Table 1Similarity $S(\tilde{A}_i, \tilde{A}_j)$ between Gaussian type-2 fuzzy sets using the proposed equation (14).

	\tilde{A}_1	\tilde{A}_2	\tilde{A}_3	\tilde{A}_4	\tilde{A}_5
\tilde{A}_1	1.0000	0.6995	0.2601	0.8735	0.1993
\tilde{A}_2	0.6995	1.0000	0.3227	0.6195	0.2630
\tilde{A}_3	0.2601	0.3227	1.0000	0.2484	0.7405
\tilde{A}_4	0.8735	0.6195	0.2484	1.0000	0.1907
\tilde{A}_5	0.1993	0.2630	0.7405	0.1907	1.0000

To determine if the sample pattern \tilde{C} is more similar to \tilde{B} than \tilde{A} , the similarity from the Hausdorff distance using the α -cut method proposed by Hung and Yang [37] is used. We see that $t_1 = 0.8$, $t_2 = 0.75$, $t_3 = 0.7$, $t_4 = 0.65$, $t_5 = 0.6$, $t_6 = 0.5$ are regarded as level values of α -cut. The Hausdorff distance [37] is calculated as follows:

$$H_f(\tilde{A}, \tilde{C}) = \frac{0.8 \times 0.2 + 0.75 \times 0.1 + 0.7 \times 0 + 0.65 \times 0.2 + 0.6 \times 0 + 0.5 \times 0}{0.8 + 0.75 + 0.7 + 0.65 + 0.6 + 0.5} = 0.1044.$$

Similarly, $H_f(\tilde{B}, \tilde{C}) = 0.1044$. According to Hung and Yang [37], the above calculated similarity values need to be converted by standarization to be $S_e(\tilde{A}, \tilde{C}) = 0.8432$ and $S_e(\tilde{B}, \tilde{C}) = 0.8432$.

Based on the above calculations, it is found that the similarity degree of \tilde{C} and \tilde{A} is the same as \tilde{C} and \tilde{B} . However, it is obvious that the sample pattern \tilde{C} is closer to the pattern \tilde{B} than \tilde{A} according to the structure of the data set $\{\tilde{A}, \tilde{B}, \tilde{C}\}$. This drawback is due to the use of α -cut method in the Hausdorff distance between type-2 fuzzy sets defined by Hung and Yang [37] where some data are not in full use and caused discrepancy.

In fact, our proposed similarity of Eq. (14) for type-2 fuzzy sets is able to improve the drawback caused by Hausdorff distance of Hung and Yang [37]. Using Eq. (14), we have that

$$\begin{aligned} S(\tilde{A}, \tilde{C}) &= \frac{\min\{0.2 \times 0.65, 0.2 \times 0.6\} + \min\{0.4 \times 0.8, 0.4 \times 0.7\} + \min\{0.5 \times 0.6, 0.5 \times 0.75\} + \min\{0.6 \times 0.5, 0.6 \times 0.9\} + \min\{0.8 \times 0.2, 0.8 \times 0.4\}}{\max\{0.2 \times 0.65, 0.2 \times 0.6\} + \max\{0.4 \times 0.8, 0.4 \times 0.7\} + \max\{0.5 \times 0.6, 0.5 \times 0.75\} + \max\{0.6 \times 0.5, 0.6 \times 0.9\} + \max\{0.8 \times 0.2, 0.8 \times 0.4\}} \\ &= 0.6884. \end{aligned}$$

Similarly, we have that $S(\tilde{B}, \tilde{C}) = 0.8607$. Thus, the sample pattern \tilde{C} is closer to the pattern \tilde{B} than \tilde{A} according to the above calculation results from our proposed similarity. This exactly matches the structure of the data set.

In final example, we combine the proposed similarity of Eq. (14) with Yang and Shih's [30] algorithm such that it can be a clustering method for type-2 fuzzy data. These clustering results will be compared with Hung and Yang's [37] results.

Example 4. Consider Gaussian type-2 fuzzy sets \tilde{A}_j , $j = 1, 2, 3, 4, 5$, with a discrete domain consisting of only 3 points, $x_1 = 1$, $x_2 = 3$ and $x_3 = 5$. Suppose that $m(x_1) = 0.1$, $m(x_2) = 0.9$ and $m(x_3) = 0.5$, $\mu_{\tilde{A}_j}(x_i) = \int_{u \in J_{x_i} \subseteq [0, 1]} \exp\{-(u - m(x_i))^2 / 2(\sigma_j^2(x_i))^2\} / u$, $i = 1, 2, 3$, $j = 1, 2, 3, 4, 5$, where $\sigma_1^2(x_i) = 1.5m(x_i)$, $\sigma_2^2(x_i) = 0.3m(x_i)$, $\sigma_3^2(x_i) = 0.02m(x_i)$, $\sigma_4^2(x_i) = 3m(x_i)$ and $\sigma_5^2(x_i) = 0.01m(x_i)$, for $i = 1, 2, 3$. We want to cluster $\{\tilde{A}_j, j = 1, 2, 3, 4, 5\}$ based on the similarity measure. The similarity degree between \tilde{A}_1 and \tilde{A}_2 using Eq. (14) is calculated as follows.

$$\begin{aligned} S(\tilde{A}_1, \tilde{A}_2) &= \frac{1}{3} \left\{ \frac{\int_0^1 \min \{u \cdot e^{-\frac{1}{2} \frac{(u-0.1)^2}{0.15}}, u \cdot e^{-\frac{1}{2} \frac{(u-0.1)^2}{0.03}}\} du}{\int_0^1 \max \{u \cdot e^{-\frac{1}{2} \frac{(u-0.1)^2}{0.15}}, u \cdot e^{-\frac{1}{2} \frac{(u-0.1)^2}{0.03}}\} du} + \frac{\int_0^1 \min \{u \cdot e^{-\frac{1}{2} \frac{(u-0.9)^2}{1.35}}, u \cdot e^{-\frac{1}{2} \frac{(u-0.9)^2}{0.27}}\} du}{\int_0^1 \max \{u \cdot e^{-\frac{1}{2} \frac{(u-0.9)^2}{1.35}}, u \cdot e^{-\frac{1}{2} \frac{(u-0.9)^2}{0.27}}\} du} \right. \\ &\quad \left. + \frac{\int_0^1 \min \{u \cdot e^{-\frac{1}{2} \frac{(u-0.5)^2}{0.75}}, u \cdot e^{-\frac{1}{2} \frac{(u-0.5)^2}{0.15}}\} du}{\int_0^1 \max \{u \cdot e^{-\frac{1}{2} \frac{(u-0.5)^2}{0.75}}, u \cdot e^{-\frac{1}{2} \frac{(u-0.5)^2}{0.15}}\} du} \right\} = 0.6995. \end{aligned}$$

By a similar calculation, we obtain the other similarities as shown in Table 1.

By examining the similarities shown in Table 1, we find that $S(\tilde{A}_1, \tilde{A}_2) = 0.8735$ and $S(\tilde{A}_3, \tilde{A}_5) = 0.7405$ are the maximum choice of similarity degrees. That is, \tilde{A}_1 and \tilde{A}_2 may be in a class, and \tilde{A}_3 and \tilde{A}_5 in another class.

The similarity measure between type-2 fuzzy sets \tilde{A}_j , $j = 1, 2, 3, 4, 5$ using Hung and Yang's [37] Hausdorff distance are presented in Table 2. By examining the similarities shown in Table 2, we find that $S_e(\tilde{A}_3, \tilde{A}_5) = 0.9810$ and $S_e(\tilde{A}_2, \tilde{A}_3) = 0.8235$ are the maximum choice of the similarity degrees.

To obtain a better profile for clustering $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4$ and \tilde{A}_5 , Yang and Shih's algorithm [25], a clustering method based on fuzzy relations by beginning with a similarity matrix, is applied to these Gaussian type-2 fuzzy sets by beginning with Tables 1 and 2.

Table 2Similarity $S_e(\tilde{A}_i, \tilde{A}_j)$ between Gaussian type-2 fuzzy sets using Hung and Yang's [37].

	\tilde{A}_1	\tilde{A}_2	\tilde{A}_3	\tilde{A}_4	\tilde{A}_5
\tilde{A}_1	1.0000	0.7169	0.5719	0.7826	0.5581
\tilde{A}_2	0.7169	1.0000	0.8235	0.5384	0.8066
\tilde{A}_3	0.5719	0.8235	1.0000	0.4134	0.9810
\tilde{A}_4	0.7826	0.5384	0.4134	1.0000	0.4014
\tilde{A}_5	0.5581	0.8066	0.9810	0.4014	1.0000

From Table 1, we get the similarity matrix $R^{(0)}$ on $\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5\}$ with

$$R_1^{(0)} = \begin{bmatrix} 1.0000 & & & & \\ 0.6995 & 1.0000 & & & \\ 0.2601 & 0.3227 & 1.0000 & & \\ 0.8735 & 0.6195 & 0.2484 & 1.0000 & \\ 0.1993 & 0.2630 & 0.7405 & 0.1907 & 1.0000 \end{bmatrix}.$$

By the max- Δ composition, we have that

$$R_1^{(0)} = R_1^{(1)} = \begin{bmatrix} 1.0000 & & & & \\ 0.6995 & 1.0000 & & & \\ 0.2601 & 0.3227 & 1.0000 & & \\ 0.8735 & 0.6195 & 0.2484 & 1.0000 & \\ 0.1993 & 0.2630 & 0.7405 & 0.1907 & 1.0000 \end{bmatrix}$$

is a max- Δ similarity relation matrix. Thus, we can obtain the following clustering results using Yang and Shih's algorithm [30]:

$$\begin{aligned} 0 < \alpha \leq 0.1907 &\Rightarrow \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5\} \\ 0.1907 < \alpha \leq 0.2484 &\Rightarrow \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4\}, \{\tilde{A}_5\} \\ 0.2484 < \alpha \leq 0.3227 &\Rightarrow \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_4\}, \{\tilde{A}_3, \tilde{A}_5\} \\ 0.3227 < \alpha \leq 0.6995 &\Rightarrow \{\tilde{A}_1, \tilde{A}_4\}, \{\tilde{A}_3, \tilde{A}_5\}, \{\tilde{A}_2\}, \\ 0.6995 < \alpha \leq 0.8735 &\Rightarrow \{\tilde{A}_1, \tilde{A}_4\}, \{\tilde{A}_2\}, \{\tilde{A}_3\}, \{\tilde{A}_5\} \\ 0.8735 < \alpha \leq 1.000 &\Rightarrow \{\tilde{A}_1\}, \{\tilde{A}_2\}, \{\tilde{A}_3\}, \{\tilde{A}_4\}, \{\tilde{A}_5\}. \end{aligned}$$

From Table 2, we get the similarity matrix $R_2^{(0)}$ on $\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5\}$ with:

$$R_2^{(0)} = \begin{bmatrix} 1.0000 & & & & \\ 0.7169 & 1.0000 & & & \\ 0.5719 & 0.8235 & 1.0000 & & \\ 0.7826 & 0.5384 & 0.4134 & 1.0000 & \\ 0.5581 & 0.8066 & 0.9810 & 0.4014 & 1.0000 \end{bmatrix}.$$

Similarly, by the max- Δ composition, we have that:

$$R_2^{(0)} = R_2^{(1)} = \begin{bmatrix} 1.0000 & & & & \\ 0.7169 & 1.0000 & & & \\ 0.5719 & 0.8235 & 1.0000 & & \\ 0.7826 & 0.5384 & 0.4134 & 1.0000 & \\ 0.5581 & 0.8066 & 0.9810 & 0.4014 & 1.0000 \end{bmatrix}$$

is a max- Δ similarity relation matrix. Thus, the following clustering results are obtained using Yang and Shih's algorithm [30]:

$$\begin{aligned} 0 < \alpha \leq 0.4014 &\Rightarrow \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5\} \\ 0.4014 < \alpha \leq 0.5384 &\Rightarrow \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_5\}, \{\tilde{A}_4\} \\ 0.5384 < \alpha \leq 0.7169 &\Rightarrow \{\tilde{A}_2, \tilde{A}_3, \tilde{A}_5\}, \{\tilde{A}_1, \tilde{A}_4\} \\ 0.7169 < \alpha \leq 0.8066 &\Rightarrow \{\tilde{A}_2, \tilde{A}_3, \tilde{A}_5\}, \{\tilde{A}_1\}, \{\tilde{A}_4\}, \\ 0.8066 < \alpha \leq 0.9810 &\Rightarrow \{\tilde{A}_3, \tilde{A}_5\}, \{\tilde{A}_1\}, \{\tilde{A}_2\}, \{\tilde{A}_4\} \\ 0.9810 < \alpha \leq 1.0000 &\Rightarrow \{\tilde{A}_1\}, \{\tilde{A}_2\}, \{\tilde{A}_3\}, \{\tilde{A}_4\}, \{\tilde{A}_5\}. \end{aligned}$$

This example shows that to each mean $m(x_i)$, comparing the variances for $\mu_{\tilde{A}_j}(x_i)$, $j = 1, 2, 3, 4, 5$, the ratio of \tilde{A}_4 variance to that of \tilde{A}_1 is 2:1, and that of \tilde{A}_3 to that of \tilde{A}_5 is 2:1 as well. It could be inferred that \tilde{A}_1 and \tilde{A}_4 may be in one class, and \tilde{A}_3

and \tilde{A}_5 can be in another class. Moreover, the ratios of \tilde{A}_1 variance to that of \tilde{A}_2 is 5:1 and that of \tilde{A}_4 to that of \tilde{A}_2 is 10:1. The ratios of \tilde{A}_2 variance to that of \tilde{A}_3 and \tilde{B}_5 are 15:1 and 30:1, respectively. Thus, \tilde{A}_2 should be closer to the class $\{\tilde{A}_1, \tilde{A}_4\}$ than the class $\{\tilde{A}_3, \tilde{A}_5\}$.

Based on the above two clustering results, our proposed similarity groups \tilde{A}_2 to the class $\{\tilde{A}_1, \tilde{A}_4\}$ and $\{\tilde{A}_3, \tilde{A}_5\}$ as another class where α -level is in $[0.2484, 0.3227]$. However, Hung and Yang [37] groups \tilde{A}_2 to the class $\{\tilde{A}_3, \tilde{A}_5\}$ and $\{\tilde{A}_1, \tilde{A}_4\}$ as another class where α -level is in $[0.5384, 0.7169]$. In summary, our proposed method actually shows more rational clustering results, compared with the results from Hung and Yang [37].

5. Conclusions

In this paper, we proposed the similarity and inclusion measures between type-2 fuzzy sets, and discussed the properties and relationships between them. For practical reasons, we presented the calculations of similarity and inclusion measures between type-2 fuzzy sets by several examples and comparisons. For an application to clustering, Yang and Shih's algorithms [30] are adopted for cluster analysis. Comparisons between the proposed results and Hung and Yang's [37] method are made. The clustering results are reasonably included in a hierarchical tree with respect to a different α -level. Because Hung and Yang [37] used α -cut, some data information might be lost when inducing the similarity between type-2 fuzzy sets so that the conclusion might be different. The proposed method in this paper actually improves the drawback of Hung and Yang's [37] method and derives more rationally hierarchical clustering results for type-2 fuzzy data. Recently, Guh et al. [38] has used fuzzy relation-based clustering [30] to establish performance evaluation structures and then applied it to evaluate the performance of various universities in Taiwan. In our further study, we will advance our results to construct type-2 relation-based clustering with its application to performance evaluation systems.

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References

- [1] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-1, *Information Sciences* 8 (1975) 199–249.
- [2] L.A. Zadeh, Fuzzy sets, *Information and Control* 8 (1965) 338–356.
- [3] M. Mizumoto, K. Tanaka, Some properties of fuzzy sets of type-2, *Information and Control* 31 (1976) 312–340.
- [4] M. Mizumoto, K. Tanaka, Fuzzy sets of type 2 under algebraic product and algebraic sum, *Fuzzy Sets and Systems* 5 (1981) 277–290.
- [5] J. Nieminen, Algebraic structure of fuzzy sets of type-2, *Kybernetika* 13 (1977) 261–273.
- [6] R.R. Yager, Fuzzy subsets of type II in decisions, *Journal of Cybernetics* 10 (1980) 137–159.
- [7] J.M. Mendel, G.C. Mouzouris, Designing fuzzy logic systems, *IEEE Transactions on Circuits and Systems-II: Analog and Digital Signal Processing* 44 (1997) 885–895.
- [8] J. Zeng, Z.Q. Liu, Type-2 fuzzy hidden Markov models and their application to speech recognition, *IEEE Transactions on Fuzzy Systems* 14 (2006) 454–470.
- [9] R.I. John, Type-2 inferencing and community transport scheduling, in: *Proc. IEEE Int. Conf. Fuzzy Systems, IEEE World Congress Intelligent Techniques Soft Computing*, Aachen, Germany, Sept. 1996, pp. 1369–1372.
- [10] H.B. Mitchell, Pattern recognition using type-II fuzzy sets, *Information Sciences* 170 (2005) 409–418.
- [11] H.B. Mitchell, Correlation coefficient for type-2 fuzzy sets, *International Journal of Intelligent Systems* 21 (2006) 143–153.
- [12] N.N. Karnik, J.M. Mendel, Applications of type-2 fuzzy logic systems to forecasting of time series, *Information Sciences* 120 (1999) 89–111.
- [13] M. Wagenknecht, K. Hartmann, Application of fuzzy sets of type-2 to the solution of fuzzy equation systems, *Fuzzy Sets and Systems* 25 (1988) 183–190.
- [14] D. Dubois, H. Prade, Operations on fuzzy numbers, *International Journal of Systems Science* 9 (1978) 613–626.
- [15] D. Dubois, H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic, New York, 1980.
- [16] N.N. Karnik, J.M. Mendel, Operations on type-2 fuzzy sets, *Fuzzy Sets and Systems* 122 (2001) 327–348.
- [17] H. Tahayori, A.G.B. Tettamanzi, G.D. Antoni, Approximated top type-2 fuzzy sets operations, in: *IEEE International Conference on Fuzzy Systems*, Sheraton Vancouver Wall Centre Hotel, Vancouver, BC, Canada, Tuly 16–21, 2006.
- [18] J.M. Mendel, R.I. John, Type-2 fuzzy sets made simple, *IEEE Transactions on Fuzzy Systems* 10 (2002) 117–127.
- [19] L.A. Zadeh, Similarity relations and fuzzy ordering, *Information Sciences* 3 (1971) 177–200.
- [20] C.P. Pappis, N.I. Karacapilidis, A comparative assessment of measures of similarity of fuzzy values, *Fuzzy Sets and Systems* 56 (1993) 171–174.
- [21] Liu Xuechang, Entropy, distance measure and similarity measure of fuzzy sets and their relations, *Fuzzy Sets and Systems* 52 (1992) 305–318.
- [22] I.B. Turksen, Z. Zhong, An approximate analogical reasoning approach based on similarity measures, *IEEE Transactions on Systems, Man and Cybernetics* 18 (1988) 1049–1056.
- [23] J.J. Buckley, Y. Hayashi, Fuzzy input–output controllers are universal approximates, *Fuzzy Sets and Systems* 58 (1993) 273–278.
- [24] K.S. Candan, W.S. Li, M.L. Priya, Similarity-based ranking and query processing in multimedia databases, *Data & Knowledge Engineering* 35 (2000) 259–298.
- [25] D. Sinha, E.R. Dougherty, Fuzzification of set inclusion: Theory and applications, *Fuzzy Sets and Systems* 55 (1993) 15–42.
- [26] V. Chatzis, I. Pitas, A generalized fuzzy mathematical morphology and its application in robust 2-D and 3-D object representation, *IEEE Transactions on Image Processing* 9 (2000) 1798–1810.
- [27] A. Kehagis, M. Konstantinidou, L-fuzzy valued inclusion measure, L-fuzzy similarity and L-fuzzy distance, *Fuzzy Sets and Systems* 136 (2003) 313–332.
- [28] C. Cornelis, C. Van der Donck, E. Kerre, Sinha–Dougherty approach to the fuzzification of set inclusion revisited, *Fuzzy Sets and Systems* 134 (2003) 283–295.
- [29] W. Zeng, H. Li, Inclusion measures, similarity measures, and the fuzziness of fuzzy sets and their relations, *International Journal of Intelligent Systems* 21 (2006) 639–653.
- [30] M.S. Yang, H.M. Shih, Cluster analysis based on fuzzy relations, *Fuzzy Sets and Systems* 120 (2001) 197–212.
- [31] R. Zwicky, E. Carlstein, D.V. Budesu, Measures of similarity among fuzzy concepts: A comparative analysis, *International Journal of Approximate Reasoning* 1 (1987) 221–242.

- [32] J. Fan, W. Xie, Some notes on similarity measure and proximity measure, *Fuzzy Sets and Systems* 101 (1999) 403–412.
- [33] M.S. Yang, W.L. Hung, S.J. Chang-Chien, On a similarity measure between LR-type fuzzy numbers and its application to database acquisition, *International Journal of Intelligent Systems* 20 (2005) 1001–1016.
- [34] P. Bosc, O. Pivert, About approximate inclusion and its axiomatization, *Fuzzy Sets and Systems* 157 (2006) 1438–1454.
- [35] L. Bloch, H. Maitre, Fuzzy mathematical morphologies: A comparative study, *Pattern Recognition* 28 (1995) 1341–1387.
- [36] M. Werman, S. Peleg, Min–max operations in texture analysis, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 7 (1985) 730–733.
- [37] W.L. Hung, M.S. Yang, Similarity measures between type-2 fuzzy sets, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 12 (2004) 827–841.
- [38] Y.Y. Guh, M.S. Yang, R.W. Po, E.S. Lee, Establishing performance evaluation structures by fuzzy relation-based cluster analysis, *Computers and Mathematics with Applications* 56 (2008) 572–582.